The quantitative difference between countable compactness and compactness

C. Angosto and B. Cascales

Universidad de Murcia

10th Prague topological symposium, Czech Republic. August 13 - 19, 2006

The papers

- B. Cascales, W. Marciszesky, and M. Raja, *Distance to spaces of continuous functions*, Topology Appl. **153** (2006), 2303–2319.
- C. Angosto and B. Cascales, *The quantitative difference between countable compactness and compactness*, Submitted, 2006.
- ______, *Distances to spaces of Baire one functions*, Work in progress, 2006.

The starting point...

List Details

- M. Fabian, P. Hájek, V. Montesinos, and V. Zizler.
  A quantitative version of Krein's Theorem..
  Rev. Mat. Iberoamericana **21** (2005), no. 1, 237–248..
- A. S. Granero.
  An extension of Krein-Šmulian theorem.
  Rev. Mat. Iberoamericana **22** (2005), no. 1, 93–110.
- A. S. Granero, P. Hájek, and V. Montesinos Santalucía.
  Convexity and w^*-compactness in Banach spaces.
  Math. Ann., 2005.

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Let E be a Banach space and let $H \subset E$ be a bounded subset of E. Then

$$\widehat{\mathrm{d}}(\overline{\mathrm{co}(H)}, E) \leq 2\widehat{\mathrm{d}}(\overline{H}, E),$$

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- $\widehat{\mathrm{d}}(A, E) := \sup\{d(a, E) : a \in A\}$ for $A \subset E^{**}$;
- $\widehat{\mathrm{d}}(A, E) = 0$ iff $A \subset E$. Hence the inequality implies Krein's theorem (if H is relatively weakly compact then $\overline{\mathrm{co}(H)}$ is weakly compact.)

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- Some of the constant involved are sharp.

...our goal

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- To take the results where *(I think!)* they belong *i.e.* to the context of $C(K)$ and $\mathbb{R}^K$ spaces endowed with τ_p;

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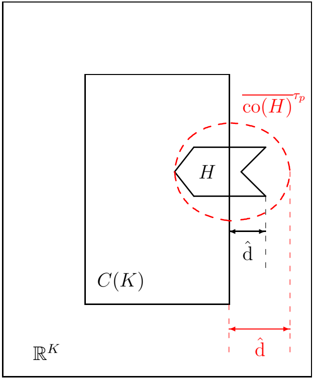

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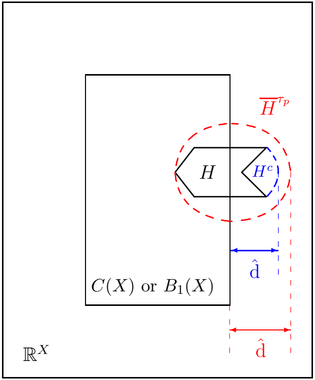

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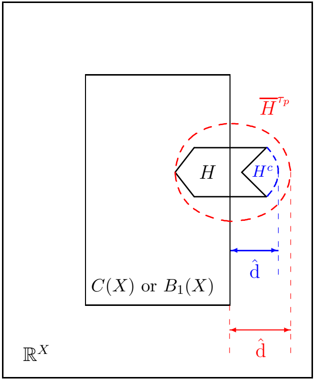

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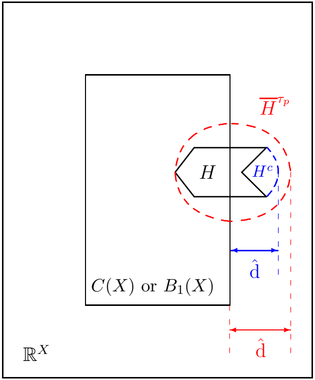

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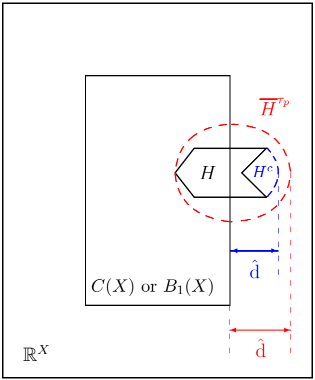

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tools

- new reading of the *classical*;
- for $C(X)$ we use *double limits* used by Grothendieck;
- for $B_1(X)$ we use the notions of *fragmentability* and *σ-fragmentability* of functions.

Quantitative Grothendieck charact. of τ_p-compactness

Theorem

If K is a compact topological space and H is a uniformly bounded subset of $C(K)$, then

$$\operatorname{ck}(H) \leq \hat{\mathrm{d}}(\overline{H}^{\mathbb{R}^K}, C(K)) \leq \gamma(H) \leq 2\operatorname{ck}(H).$$

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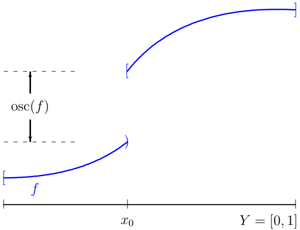

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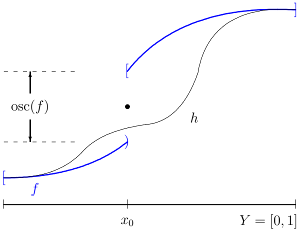

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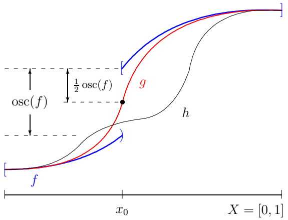

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$$\alpha > \mathrm{ck}(H) = \sup_{(h_n)_n \subset H} d\Big(\bigcap_{m\in\mathbb{N}} \overline{\{h_n : n > m\}}^{\mathbb{R}^K}, C(K)\Big) \geq d\Big(\bigcap_{m\in\mathbb{N}} \overline{\{f_n : n > m\}}^{\mathbb{R}^K}, C(K)\Big)$$

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Theorem

If K is a compact topological space and H be a uniformly bounded subset and a uniformly bounded subset H of $\mathbb{R}^K$ we have that

$$\gamma(H) = \gamma(\mathrm{co}(H)),$$

and as a consequence we obtain for $H \subset C(K)$ that

$$\hat{\mathrm{d}}(\overline{\mathrm{co}(H)}^{\mathbb{R}^K}, C(K)) \leq 2\hat{\mathrm{d}}(\overline{H}^{\mathbb{R}^K}, C(K)). \tag{1}$$

and in the general case $H \subset \mathbb{R}^K$

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2. When $H \subset \mathbb{R}^K$, we approximate H by some set in $C(K)$, then use (1) and 5 appears as a simple

$$5 = 2 \times 2 + 1.$$

The results for $C(X)$

If X is a topological space, (Z,d) a metric space and H a relatively compact subset of the space (Z^X,τ_p) we define

$$\mathrm{ck}(H) := \sup_{(h_n)_n \subset H} d\Big(\bigcap_{m\in\mathbb{N}} \overline{\{h_n : n > m\}}^{Z^X}, C(X,Z)\Big).$$

Theorem

Let X be a countably K-determined space, (Z,d) a separable metric space and H a relatively compact subset of the space (Z^X,τ_p). Then

$$\mathrm{ck}(H) \overset{(a)}{\leq} \hat{d}(\overline{H}^{Z^X}, C(X,Z)) \overset{(b)}{\leq} 3\,\mathrm{ck}(H) + 2\hat{d}(H, C(X,Z)) \overset{(c)}{\leq} 5\,\mathrm{ck}(H).$$

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For the particular case $\operatorname{ck}(H)=0$ we obtain all known results about compactness in $C_p(X)$ spaces.

The technicalities for $C(X)$

Definition

Let (Z,d) be a metric space, X a set and $\varepsilon \geq 0$.

(i) We say that a sequence $(f_m)_m$ in Z^X ε-interchanges limits with a sequence $(x_n)_n$ in X if whenever the limits below exist we have

$$d(\lim_n \lim_m f_m(x_n), \lim_m \lim_n f_m(x_n)) \leq \varepsilon.$$

(ii) We say that a subset H of Z^X ε-interchanges limits with a subset A of X, if each sequence in H ε-interchanges limits with each sequence in A.

X topological space, (Z,d) a separable metric space and $H \subset (Z^X, \tau_p)$ relatively compact.

Lemma 1

If we define $\varepsilon := \mathrm{ck}(H) + \hat{d}(H, C(X,Z))$, then H 2ε-interchanges limits with relatively countably compact subsets of X.

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Lemma 2

(i) there is $\Sigma \subset \mathbb{N}^{\mathbb{N}}$ and a family $\{A_\alpha : \alpha \in \Sigma\}$ of non-void subsets of the set X such that $X = \bigcup\{A_\alpha : \alpha \in \Sigma\}$;

(ii) for every $\alpha = (a_1, a_2, \dots) \in \Sigma$ the set H ε-interchanges limits in Z with every sequence $(x_n)_n$ in X that is eventually in each set $C_{\alpha|m}$, $m \in \mathbb{N}$, where $C_{\alpha|m} = \bigcup\{A_\beta : \beta \in \Sigma \text{ and } \beta|m = \alpha|m\}$.

Then for any $f \in \overline{H}^{Z^X}$ there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in H such that

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2. Take $A_\alpha := T(\alpha)$ for every $\alpha \in \Sigma$: (i) in Lemma 2 is satisfied.
3. For every $\alpha \in \Sigma$, every sequence $(x_n)_n$ in X that is eventually in each set $C_{\alpha|m}$, $m \in \mathbb{N}$, lies in a compact subset of X.

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Let X be a countably K-determined space. Then, for any $f \in \overline{H}^{Z^X}$ there exists a sequence $(f_n)_n$ in H such that

$$\sup_{x\in X} d(g(x), f(x)) \overset{(a)}{\leq} 2\,\mathrm{ck}(H) + 2\hat{d}(H, C(X,Z)) \overset{(b)}{\leq} 4\,\mathrm{ck}(H)$$

for any cluster point g of (f_n) in Z^X.

Proof.-

1. Let $T : \Sigma \to 2^X$ be the *usco* map, $\Sigma \subset \mathbb{N}^{\mathbb{N}}$, such that $\bigcup\{T(\alpha) : \alpha \in \Sigma\} = X$;
2. Take $A_\alpha := T(\alpha)$ for every $\alpha \in \Sigma$: (i) in Lemma 2 is satisfied.
3. For every $\alpha \in \Sigma$, every sequence $(x_n)_n$ in X that is eventually in each set $C_{\alpha|m}$, $m \in \mathbb{N}$, lies in a compact subset of X.
4. Apply Lemma 1 to obtain that for

$$\varepsilon := \mathrm{ck}(H) + \hat{d}(H, C(X,Z))$$

(ii) in Lemma 2 is satisfied.

X topological space, (Z,d) a separable metric space and $H \subset (Z^X, \tau_p)$ relatively compact.

Lemma 1

If we define $\varepsilon := \mathrm{ck}(H) + \hat{d}(H, C(X,Z))$, then H 2ε-interchanges limits with relatively countably compact subsets of X.

Lemma 2

(i) there is $\Sigma \subset \mathbb{N}^{\mathbb{N}}$ and a family $\{A_\alpha : \alpha \in \Sigma\}$ of non-void subsets of the set X such that $X = \bigcup\{A_\alpha : \alpha \in \Sigma\}$;

(ii) for every $\alpha = (a_1, a_2, \dots) \in \Sigma$ the set H ε-interchanges limits in Z with every sequence $(x_n)_n$ in X that is eventually in each set $C_{\alpha|m}$, $m \in \mathbb{N}$, where $C_{\alpha|m} = \bigcup\{A_\beta : \beta \in \Sigma \text{ and } \beta|m = \alpha|m\}$.

Then for any $f \in \overline{H}^{Z^X}$ there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in H such that

$$\sup_{x\in X} d(g(x), f(x)) \le \varepsilon$$

for any cluster point g of $(f_n)_{n\in\mathbb{N}}$ in Z^X.

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4. Apply Lemma 1 to obtain that for
$$\varepsilon := \mathrm{ck}(H) + \hat{d}(H, C(X,Z))$$
(ii) in Lemma 2 is satisfied.
5. Lemma 2 finishes the proof.

The results for $C(X)$

If X is a topological space, (Z,d) a metric space and H a relatively compact subset of the space (Z^X,τ_p) we define

$$\operatorname{ck}(H) := \sup_{(h_n)_n\subset H} d\Big(\bigcap_{m\in\mathbb{N}} \overline{\{h_n : n > m\}}^{Z^X}, C(X,Z)\Big).$$

Theorem

Let X be a countably K-determined space, (Z,d) a separable metric space and H a relatively compact subset of the space (Z^X,τ_p). Then, for any $f\in\overline{H}^{Z^X}$ there exists a sequence $(f_n)_n$ in H such that

$$\sup_{x\in X} d(g(x),f(x)) \overset{(a)}{\leq} 2\operatorname{ck}(H) + 2\hat{d}(H,C(X,Z)) \overset{(b)}{\leq} 4\operatorname{ck}(H)$$

for any cluster point g of (f_n) in Z^X.

Theorem

Let X be a countably K-determined space, (Z,d) a separable metric space and H a relatively compact subset of the space (Z^X,τ_p). Then

$$\operatorname{ck}(H) \overset{(a)}{\leq} \hat{d}(\overline{H}^{Z^X}, C(X,Z)) \overset{(b)}{\leq} 3\operatorname{ck}(H) + 2\hat{d}(H,C(X,Z)) \overset{(c)}{\leq} 5\operatorname{ck}(H).$$

For the particular case $\operatorname{ck}(H)=0$ we obtain angelicity of $C_p(X)$ (Orihuela).

If K is a compact convex subset of a l.c.s., $\mathscr{A}(K)$ is the space of affine functions defined on K, and $\mathscr{A}^C(K) = C(K) \cap \mathscr{A}(K)$.

Theorem

Let K be a compact convex subset of a l.c.s. Then for any bounded function f in $\mathscr{A}(K)$ we have

$$d(f, C(K)) = d(f, \mathscr{A}^C(K)).$$

Corollary

Let E be a Banach space and let B_{E^*} be the closed unit ball in the dual E^* endowed with the w^*-topology. Let $i : E \to E^{**}$ and $j : E^{**} \to \ell_\infty(B_{E^*})$ be the canonical embedding. Then, for every $x^{**} \in E^{**}$ we have:

$$d(x^{**}, i(E)) = d(j(x^{**}), C(B_{E^*})).$$

Measures of weak noncompactness

Definition

Given a bounded subset H of a Banach space E we define:

$$\omega(H) := \inf\{\varepsilon > 0 : H \subset K_\varepsilon + \varepsilon B_E \text{ and } K_\varepsilon \subset X \text{ is } w\text{-compact}\},$$

$$\gamma(H) := \sup\{|\lim_n \lim_m f_m(x_n) - \lim_m \lim_n f_m(x_n)| : (f_m) \subset B_{E^*}, (x_n) \subset H\},$$

assuming the involved limits exist,

$$\mathrm{ck}(H) := \sup_{(h_n)_n \subset H} d\Big(\bigcap_{m\in\mathbb{N}} \overline{\{h_n : n > m\}}^{w^*}, E\Big),$$

$$\mathrm{k}(H) := \hat{d}(\overline{H}^{w^*}, E) = \sup_{x^{**}\in\overline{H}^{w^*}} d(x^{**}, E),$$

where the w^*-closures are taken in E^{**} and the distance d is the usual inf distance for sets associated to the natural norm in E^{**}.

Relationship between measures of weak noncompactness

Theorem

For any bounded subset H of a Banach space E we have:

$$\mathrm{ck}(H) \leq \mathrm{k}(H) \leq \gamma(H) \leq 2\,\mathrm{ck}(H) \leq 2\,\mathrm{k}(H) \leq 2\omega(H),$$

$$\gamma(H) = \gamma(\mathrm{co}(H)) \text{ and } \omega(H) = \omega(\mathrm{co}(H)).$$

For any $x^{**} \in \overline{H}^{w^*}$, there is a sequence $(x_n)_n$ in H such that

$$\|x^{**} - y^{**}\| \leq \gamma(H)$$

for any cluster point y^{**} of $(x_n)_n$ in E^{**}. Furthermore, H is weakly relatively compact in E if, and only if, one (equivalently all) of the numbers $\mathrm{ck}(H), \mathrm{k}(H), \gamma(H)$ and $\omega(H)$ is zero.

Remark

The result above is the quantitative version of Eberlein-Smulyan and Krein-Smulyan theorems. From $\mathrm{k}(\mathrm{co}(H)) \leq 2\,\mathrm{k}(H)$ straightforwardly follows Krein-smulyan theorem.

Corson property implies $\mathrm{k}(\cdot) = \mathrm{ck}(\cdot)$

Theorem

If E is a Banach space with Corson property $\mathscr{C}$, then for every bounded set $H \subset E$ we have $\mathrm{ck}(H) = \mathrm{k}(H)$.

Problem

Do we have the equality $\mathrm{ck}(\cdot) = \mathrm{k}(\cdot)$ for every Banach space?

Other applications to Banach spaces

Theorem (Grothendieck)

Let K be a compact space and let H be a uniformly bounded subset of $C(K)$. Let us define

$$\gamma_K(H) := \sup\{|\lim_n \lim_m f_m(x_n) - \lim_m \lim_n f_m(x_n)| : (f_m) \subset H, (x_n) \subset K\},$$

assuming the involved limits exist. Then we have

$$\gamma_K(H) \leq \gamma(H) \leq 2\gamma_K(H).$$

Theorem (Gantmacher)

Let E and F be Banach spaces, $T : E \to F$ an operator and $T^* : F^* \to E^*$ its adjoint. Then

$$\gamma(T(B_E)) \leq \gamma(T^*(B_{F^*})) \leq 2\gamma(T(B_E)).$$

Other applications to Banach spaces

Remark: Astala and Tylli [AT90, Theorem 4]

There is separable Banach space E and a sequence $(T_n)_n$ of operators $T_n : E \to c_0$ such that

$$\omega(T_n^*(B_{\ell^1})) = 1 \qquad \text{and} \qquad \omega(T_n^{**}(B_E^{**})) \leq w(T_n(B_E)) \leq \frac{1}{n}.$$

Note that this example says, in particular, that there are no constants $m, M > 0$ such that for any bounded operator $T : E \to F$ we have

$$m\omega(T(B_E)) \leq \omega(T^*(B_{F^*})) \leq M\omega(T(B_E)).$$

Corollary

γ and ω are not equivalent measures of weak noncompactness, namely there is no $N > 0$ such that for any Banach space and any bounded set $H \subset E$ we have

$$\omega(H) \leq N\gamma(H).$$

How to measure distances to $B_1(X)$?

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We use an index of σ-fragmentability.

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If X topological space, (Z,d) a metric and $f \in Z^X$ and $\varepsilon > 0$:

1. f is ε-fragmented if for every non empty subset $F \subset X$ there exist an open subset $U \subset X$ such that $U \cap F \neq \emptyset$ and $\operatorname{diam}(f(U \cap F)) \leq \varepsilon$;

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2. f is $\varepsilon - \sigma$-fragmented by *closed sets* if there is countable family of closed subsets $(X_n)_n$ that covers X such that $f|_{X_n}$ is ε-fragmented for every $n \in \mathbb{N}$.

Indexes of fragmentability and σ-fragmentability

Definition

If X topological space, (Z,d) a metric and $f \in Z^X$. We define:

$$\sigma\text{-frag}_c(f) := \inf\{\varepsilon > 0 : f \text{ is } \varepsilon - \sigma\text{-fragmented by closed sets}\}$$

Theorem

If X is a metric space, E a Banach space and $f \in E^X$ then

$$\frac{1}{2}\sigma\text{-frag}_c(f) \leq d(f, B_1(X,E)) \leq \sigma\text{-frag}_c(f).$$

In the particular case $E = \mathbb{R}$ we precisely have

$$d(f, B_1(X)) = \frac{1}{2}\sigma\text{-frag}_c(f).$$

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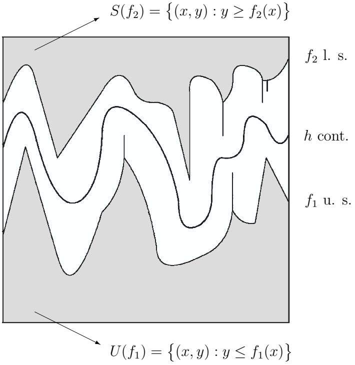
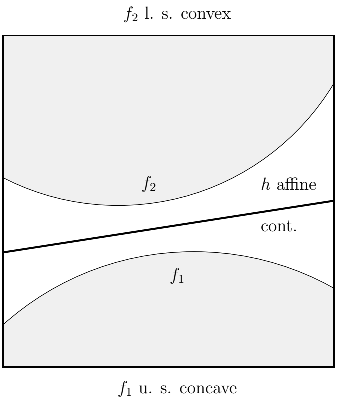


Katetov theorem (X normal)



Hahn